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TWO UNIVERSAL 3-QUANTIFIER REPRESENTATIONS OF RECURSIVELY ENUMERABLE SETS

1. We shall use the following notation: Lower-case Latin letters from a to n (inclusive) with or without subscripts will be used as variables for nonnegative integers, the remaining lower-case Latin letters will be used as variables for integers. Analogously, lower case Greek letters from α to ν will be used as metavariables for nonnegative integers, and the remaining the Greek letters will be used as metavariables for integers.

Upper case Latin letters will denote polynomials. Here and below it is to be understood that only polynomials with integer coefficients are being considered.

2. We say that a set \mathfrak{R} of nonnegative integers is represented by an arithmetic formula \mathfrak{F} with one free variable a if the equivalence $a \in \mathfrak{R} \Leftrightarrow \mathfrak{F}$ is true.

As K. Gödel showed, any recursively enumerable set is represented by an arithmetic formula. One can improve this result by restricting the kinds of formulas in various ways. Such restricted representations were investigated in [2-13]. The aim of this paper is to show that every recursively enumerable

set is represented by formulas of each of the two kinds following:

$$\exists b \exists c \bigwedge_{i=1}^{\varepsilon} \exists d [P_i(a, b, c) < D_i(a, b, c) d < Q_i(a, b, c)], \quad (1)$$

$$\exists b \exists c \forall f [f \leq F(a, b, c) \Rightarrow W(a, b, c, f) > 0]. \quad (2)$$

3. Let \mathfrak{R} be a recursively enumerable set of non-negative integers. We begin with a formula that represents the set \mathfrak{R} of the form

$$\exists h_1 \dots \exists h_\delta [R(a, h_1, \dots, h_\delta) = 0], \quad (3)$$

(the existence of such a formula is proved, for example, in [6-9,14]).

Denoting the degree of the polynomial R by λ , without loss of generality we may assume that $\lambda \geq 1$.

In order that formula (2) be equivalent to formula (3), the pair $\langle b, c \rangle$, whose existence is asserted in (2), must carry all of the information contained in the δ -tuple $\langle h_1, \dots, h_\delta \rangle$, whose existence is asserted in (3).

Many methods are known for coding tuples of nonnegative integers using a single nonnegative integer or a pair of such integers. The rather unusual method that we use allows us to check the truth of the relation

$$R(a, h_1, \dots, h_\delta) = 0 \quad (4)$$

directly from the code, without first finding the individual numbers h_1, \dots, h_δ . We define $B(h_1, \dots, h_\delta, k)$ to be the polynomial

$$\sum_{i=1}^{\delta} h_i k^{(\lambda+1)^i}.$$

This polynomial has the “geometric” interpretation: if k is greater than each of the numbers $1, h_1, \dots, h_\delta$, then h_1, \dots, h_δ are the corresponding $(\lambda+1)$ -th, \dots , $(\lambda+1)^\delta$ -th digits of the number $B(h_1, \dots, h_\delta, k)$ in the k -ary number system, while all of the other digits are zeros.

One can easily verify that an identity of the following type holds:

$$(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda = \sum_{\alpha_0 + \dots + \alpha_\delta \leq \lambda} \kappa_{\alpha_0, \dots, \alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} k^{N(\alpha_0, \dots, \alpha_\delta)} \quad (5)$$

where

$$N(l_0, \dots, l_\delta) = \sum_{\iota=0}^{\delta} l_\iota (\lambda + 1)^\iota,$$

and $\kappa_{\alpha_0, \dots, \alpha_\delta}$ are positive integers. One can easily see that this polynomial N has the following property:

$$\left\{ \&_{\iota=0}^{\delta} ((l'_\iota \leq \lambda) \& (l''_\iota \leq \lambda) \& N(l'_0, \dots, l'_\delta) = N(l''_0, \dots, l''_\delta)) \right\} \Rightarrow \&_{\iota=0}^{\delta} (l'_\iota = l''_\iota). \quad (6)$$

This property is obvious applying the above mentioned “geometric” interpretation to the polynomial N : if $l_\iota \leq \lambda$, then l_δ, \dots, l_0 are precisely the digits in the expansion of $N(l_0, \dots, l_\delta)$ in the number system with base $\lambda + 1$. The polynomial B was chosen in such a way that in the k -ary expansion of the number $B(h_1, \dots, h_\delta, k)$ the non-zero digits are placed in special locations in order to obtain property (6). Property (6) allows us to give “a geometric interpretation” of identity (5): in the k -ary expansion of the number $(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda$ the digits are all possible numbers of the form

$$\kappa_{\alpha_0, \dots, \alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} \quad (7)$$

provided that k exceeds each of them.

Without loss of generality we shall assume that the polynomial R is a linear combination of monomials (7):

$$R(a, h_1, \dots, h_\delta) = \sum_{\alpha_0 + \dots + \alpha_\delta \leq \lambda} \rho_{\alpha_0, \dots, \alpha_\delta} \kappa_{\alpha_0, \dots, \alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} \quad (8)$$

Obviously, if $l_1 + \dots + l_\delta \leq \lambda$, then

$$N(l_1, \dots, l_\delta) \leq \lambda(\lambda + 1)^\delta.$$

Let us denote $\lambda(\lambda + 1)^\delta$ by ν , and the polynomial

$$\sum_{\alpha_0 + \dots + \alpha_\delta \leq \lambda} \rho_{\alpha_0, \dots, \alpha_\delta} k^{\nu - N(\alpha_0, \dots, \alpha_\delta)}$$

by $V(k)$. One can easily see that an identity of the following type holds:

$$V(k)(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda = \sum_{\iota=0}^{2\nu} T_\iota(a, h_1, \dots, h_\delta) k^\iota, \quad (9)$$

where $T_0, \dots, T_{2\nu}$ are polynomials whose degrees do not exceed λ .

One can interpret identity (9) in a natural manner if one considers a k -ary number system in which negative digits are allowed; for instance, one may require that

$$k > |2T_\iota(a, h_1, \dots, h_\delta)|, \quad (\iota = 0, \dots, 2\nu) \quad (10)$$

and consider the system with digits ranging from $[-(k-1)/2]$ to $[(k-1)/2]$.

It is easy to check that (5), (6), (8) and (9) imply the identity

$$T_\nu(a, h_1, \dots, h_\delta) = R(a, h_1, \dots, h_\delta). \quad (11)$$

Thus, if

$$b = B(h_1, \dots, h_\delta, k) \quad (12)$$

and k is sufficiently large that the inequalities (10) are satisfied, then the relation (4) holds if and only if the digit in the ν -th place in the k -ary expansion of the number $V(k)(1 + ak + b)^\lambda$ is zero. As we shall show below, the latter condition can be easily written using a single existential quantifier.

Lemma 1. For any $a, b, h_1, \dots, h_\delta, k$ satisfying conditions (10) and (12), the relation (4) holds if and only if there exists an integer z such that

$$-k^\nu < 2(V(k)(1 + ak + b)^\lambda - zk^{\nu+1}) < k^\nu. \quad (13)$$

Necessity. Put

$$z = \sum_{\iota=\nu+1}^{2\nu} T_\iota(a, h_1, \dots, h_\delta) k^{\iota-\nu-1}$$

By (9), (12) and (4),

$$V(k)(1 + ak + b)^\lambda - zk^{\nu+1} = \sum_{\iota=0}^{\nu-1} T_\iota(a, h_1, \dots, h_\delta) k^\iota.$$

We deduce from (10) that

$$\begin{aligned}
\left| 2 \sum_{\iota=0}^{\nu-1} T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota} \right| &\leq \\
&\leq \sum_{\iota=0}^{\nu-1} |2T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota}| \leq \\
&\leq \sum_{\iota=0}^{\nu-1} (k-1) k^{\iota} = k^{\nu} - 1 < k^{\nu}, \quad (14)
\end{aligned}$$

so that inequalities (13) are satisfied.

Sufficiency. It is easy to see that there exists at most one integer y such that

$$-k^{\nu} < 2(V(k)(1 + ak + b)^{\lambda} - yk^{\nu}) < k^{\nu}.$$

On the one hand, by (13), y equals zk , while on the other hand y equals

$$\sum_{\iota=\nu}^{2\nu} T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota-\nu}$$

since, by (9) and (12),

$$\begin{aligned}
V(k)(1 + ak + b)^{\lambda} - \left(\sum_{\iota=\nu}^{2\nu} T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota-\nu} \right) k^{\nu} &= \\
&= \sum_{\iota=0}^{\nu-1} T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota}
\end{aligned}$$

and inequalities (14) hold. Thus,

$$\sum_{\iota=\nu}^{2\nu} T_{\iota}(a, h_1, \dots, h_{\delta}) k^{\iota-\nu} = zk.$$

Passing from this equation to a congruence, we get

$$T_{\nu}(a, h_1, \dots, h_{\delta}) \equiv 0 \pmod{k}.$$

This, together with (10), gives us the equality

$$T_{\nu}(a, h_1, \dots, h_{\delta}) = 0. \quad (15)$$

Now (4) follows from (11) and (15).

The Lemma is proved.

4. We now proceed to transform the inequalities (10). Let γ be a positive integer that exceeds twice the sum of the absolute values of the coefficients of all the polynomial $T_1, \dots, T_{2\nu}$. Obviously, the following inequalities hold:

$$|2T_i(a, h_1, \dots, h_\delta)| < \gamma(\max\{1, a, h_1, \dots, h_\delta\})^\lambda.$$

Thus, if

$$c \geq \max\{h_1, \dots, h_\delta\}$$

and

$$k = K(a, c) = \gamma(2 + a + c)^\lambda,$$

then inequalities (10) are satisfied.

Using Lemma 1 one can easily show that formula (3) is equivalent to the formula

$$\exists b \exists c [\mathfrak{F}_1 \ \& \ \exists z \mathfrak{F}_2],$$

where, here and below \mathfrak{F}_1 denotes the formula

$$\exists h_1 \dots \exists h_\delta [c \geq \max\{h_1, \dots, h_\delta\} \ \& \ b = H(h_1, \dots, h_\delta, K(a, c))]$$

and \mathfrak{F}_2 denotes the formula

$$\begin{aligned} - (K(a, c))^\nu &< 2(V(K(a, c))(1 + aK(a, c) + b)^\lambda - \\ &- zK(a, c)^{\nu+1}) < (K(a, c))^\nu. \end{aligned}$$

Lemma 2. Formula \mathfrak{F}_1 is equivalent to the formula

$$\mathfrak{F}_3 \ \& \ \mathfrak{F}_4 \ \& \ \mathfrak{F}_5, \tag{16}$$

where, here and below \mathfrak{F}_3 denotes the formula

$$\exists d [b = d(K(a, c))^{\lambda+1}],$$

\mathfrak{F}_4 denotes the formula

$$\begin{aligned} &\overset{\delta-1}{\underset{\iota=1}{\&}} \exists d \exists e [b = d(K(a, c))^{(\lambda+1)^\iota+1} + e \ \& \ e < (c+1)(K(a, c))^{(\lambda+1)^\iota}], \end{aligned}$$

and \mathfrak{F}_5 denotes the formula

$$b < (c + 1)(K(a, c))^{(\lambda+1)^\delta}.$$

The truth of this lemma becomes quite clear, if one notes that each of the formulas \mathfrak{F}_1 and (16) mean that in the expansion of the number b in the number system with the base $K(a, c)$ the non-zero digits can only occupy the $(\lambda + 1)$ -th, \dots , $(\lambda + 1)^\delta$ -th positions, and moreover, these digits do not exceed c .

5. Combining Lemmas 1 and 2. we see that formula (3) is equivalent to the formula

$$\exists b \exists c [\exists z \mathfrak{F}_2 \ \& \ \mathfrak{F}_3 \ \& \ \mathfrak{F}_4 \ \& \ \mathfrak{F}_5]. \quad (17)$$

Theorem 1. Every recursively enumerable set of nonnegative integers can be represented by a formula of the form (1).

Proof. The desired formula can be obtained from the formula (17) by means of easy algebraic transformations.

Formula \mathfrak{F}_2 contains the variable z , whose possible values are all integers. However, it follows from \mathfrak{F}_2 that

$$2z(K(a, c))^{\nu+1} > -(K(a, c))^\nu + 2V(K(a, c))(1 + aK(a, c) + b)^\lambda,$$

so that

$$z \geq V(K(a, c))(1 + aK(a, c) + b)^\lambda.$$

Let us denote by \mathfrak{F}_6 the formula, which is obtained from \mathfrak{F}_2 by substituting the polynomial

$$d + V(K(a, c))(1 + aK(a, c) + b)^\lambda$$

for z and by transposing terms, which do and do not contain d to opposite sides of the inequalities. Obviously, the formula $\exists z \mathfrak{F}_2$ is equivalent to the formula $\exists d \mathfrak{F}_6$.

We transform the formula \mathfrak{F}_3 into the equivalent formula

$$\exists d [b - 1 < (K(a, c))^{\lambda+1} d < b + 1]. \quad (18)$$

Each of the conjuncts composing the formula \mathfrak{F}_4 includes an equation that enable us to express e explicitly in terms of a , b , c and d , and, therefore, to

eliminate this variable. In addition, we must impose an inequality to insure the non-negativity of e . Finally, we obtain the formula

$$\bigwedge_{\iota=1}^{\delta-1} \exists d [b - (c+1)(K(a, c))^{(\lambda+1)^\iota} < (K(a, c))^{(\lambda+1)^{\iota-1}} d < b+1]. \quad (19)$$

Finally we must replace the formula \mathfrak{F}_5 by the equivalent formula

$$\exists d [b-1 < bd < (c+1)(K(a, c))^{(\lambda+1)^\delta}].$$

The theorem is proved.

6. Now we turn to constructing a formula of the form (2) that represents the set \mathfrak{R} . For this purpose we first show that for any ϵ there exist polynomials F_ϵ and W_ϵ in 2ϵ and $3\epsilon+1$ variables respectively, such that: if the numbers $g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon$ satisfy the inequalities

$$0 < g_\iota, \quad t_\iota - s_\iota \leq g_\iota \quad (\iota = 1, \dots, \epsilon), \quad (20)$$

then the formula

$$\bigwedge_{\iota=1}^{\epsilon} \exists z [s_\iota < zg_\iota < t_\iota] \quad (21)$$

is equivalent to the formula

$$\forall f [f \leq F_\epsilon(s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon) \Rightarrow W_\epsilon(g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, f) > 0].$$

We start with the case $\epsilon = 1$ and find, to begin with, polynomials X and Y such that for $g > 0$ the formula

$$\exists z [s < zg < t] \quad (22)$$

is equivalent to the formula

$$\forall y [-s^2 - t^2 - 2 < y \leq s^2 + t^2 + 2 \Rightarrow X(g, s, t, y) > 0 \vee Y(g, s, t, y) > 0]. \quad (23)$$

Lacking existential quantifiers, formula (23) must somehow contain complete information about an integer z that satisfies the inequalities

$$s < zg < t. \quad (24)$$

We shall verify equivalence between formulas of the forms (23) and (22) by means of the following obvious lemma, which may be regarded as a discrete analogue of the Cauchy theorem about the vanishing of a continuous function, whose values at the endpoints of an interval have opposite signs.

Let p and q be integers such that $p < q$, let Φ and Ψ be unary predicates defined for all integers between p and q . If $\Phi(p) \& \Psi(q)$ holds and for any w , such that $p < w < q$, $\Phi(w) \vee \Psi(w)$ holds, then there exists an integer r such that $p \leq r \leq q$ and $\Phi(r) \& \Psi(r+1)$.

Lemma 3. If

$$g > 0, \tag{25}$$

then formula (22) is equivalent to the formula

$$\forall y[-s^2 - t^2 - 2 < y \leq s^2 + t^2 + 2 \Rightarrow (y-1)g - s > 0 \vee t - yg > 0]. \tag{26}$$

Proof. Let g, s, t satisfy conditions (25) and (26). We will show that they satisfy condition (22), as well.

By (25),

$$\begin{aligned} t - (-s^2 - t^2 - 1)g &\geq t + s^2 + t^2 + 1 > 0, \\ (s^2 + t^2 + 1)g - s &\geq s^2 + t^2 + 1 - s > 0. \end{aligned}$$

By the discrete analogue of the Cauchy theorem mentioned above, we have that there exists z such that

$$t - zg > 0 \& zg - s > 0.$$

Thus, condition (22) is satisfied.

Now, let g, s and t satisfy conditions (25) and (22). We will find a z that satisfies inequalities (24). Suppose that condition (26) doesn't hold. Let y be a number such that

$$(y-1)g - s \leq 0 \& t - yg \leq 0. \tag{27}$$

From (24) and (27) we obtain

$$(y-1)g \leq s < zg, \quad zg < t < yg.$$

Consequently

$$y - 1 < z < y.$$

This contradiction completes the proof of the equivalence of formulas (22) and (26).

Note, that if

$$t - s \leq g, \quad (28)$$

then two inequalities in formula (27) are inconsistent. Moreover, if $(y - 1)g - s > 0$, then $t - yg < 0$, and conversely if $t - yg > 0$, then $(y - 1)g - s < 0$. This enables us to transform the disjunction of a pair of inequalities into a single one:

$$\begin{aligned} (y - 1)g - s > 0 \vee t - yg > 0 &\Leftrightarrow ((y - 1)g - s > 0 \& t - yg < 0) \vee \\ &(t - yg > 0 \& (y - 1)g < 0) \Leftrightarrow ((y - 1)g - s)(yg - t) > 0. \end{aligned}$$

Thus, if inequalities (25) and (28) are satisfied, then formula (22) is equivalent to the formula

$$\forall y[-s^2 - t^2 - 2 < y \leq s^2 + t^2 + 2 \Rightarrow Z(g, s, t, y) > 0],$$

where, here and below $Z(g, s, t, y)$ denotes the polynomial

$$((y - 1)g - s)(yg - t).$$

Note, that if $g > 0$, then

$$\forall y[y \leq -s^2 - t^2 - 2 \vee y > s^2 + t^2 + 2 \Rightarrow Z(g, s, t, y) > 0]. \quad (29)$$

7. Now consider an arbitrary formula of the form (21). If the numbers $g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon$ satisfy inequalities (20), then, as shown above, formula (21) is equivalent to the formula

$$\bigwedge_{\iota=1}^{\epsilon} \forall y[-s_\iota^2 - t_\iota^2 - 2 < y \leq s_\iota^2 + t_\iota^2 + 2 \Rightarrow Z(g_\iota, s_\iota, t_\iota, y) > 0]. \quad (30)$$

We introduce the following notation:

$$F_\iota(s_1, \dots, s_\iota, t_1, \dots, t_\iota) = \sum_{\mu=1}^{\iota} (2s_\mu^2 + 2t_\mu^2 + 4) \quad (\iota = 0, \dots, \epsilon),$$

$$Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y) = \\ Z(g_\iota, s_\iota, t_\iota, y - F_{\iota-1}(s_1, \dots, s_{\iota-1}, t_1, \dots, t_{\iota-1}) - s_\iota^2 - t_\iota^2 - 2) \quad (\iota = 1, \dots, \epsilon).$$

Obviously, formula (30) is equivalent to the formula

$$\begin{aligned} \bigwedge_{\iota=1}^{\epsilon} \forall y [F_{\iota-1}(s_1, \dots, s_{\iota-1}, t_1, \dots, t_{\iota-1}) < y \leq \\ \leq F_\iota(s_1, \dots, s_\iota, t_1, \dots, t_\iota) \Rightarrow \\ \Rightarrow Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y) > 0]. \end{aligned} \quad (31)$$

Let us denote by $W_\epsilon(g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, y)$ the polynomial

$$\prod_{\iota=1}^{\epsilon} Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y).$$

Lemma 4. If the numbers $g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon$ satisfy inequalities (20), then formula (21) is equivalent to the formula

$$\begin{aligned} \forall f [f \leq F_\epsilon(s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon) \Rightarrow \\ \Rightarrow W_\epsilon(g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, f) > 0]. \end{aligned}$$

One can easily carry out the proof of the lemma using property (29).

Theorem 2. Every recursively enumerable set of non-negative integers can be represented by a formula of the form (2).

Proof. We will transform the formula $\exists z \mathfrak{F}_2 \& \mathfrak{F}_3 \& \mathfrak{F}_4 \& \mathfrak{F}_5$ into a form analogous to (21).

In the formula \mathfrak{F}_2 it suffices to transpose terms, which do or do not contain z , to opposite sides of the inequalities. We denote the resulting formula by \mathfrak{F}_7 .

In the formula \mathfrak{F}_3 we replace the variable d , whose admissible values are nonnegative integers, by the variable z , whose admissible values are all integers. Since

$$b \geq 0, \quad (K(a, c))^{\lambda+1} > 0,$$

the formula thus obtained is equivalent to the formula \mathfrak{F}_3 . Rewriting the formula we obtained in a form analogous to (18), we denote the new formula by \mathfrak{F}_8 .

Analogously, in each conjunct of the formula \mathfrak{F}_4 we replace the variable d by z . Since always

$$b \geq 0, \quad (K(a, c))^{(\lambda+1)^{e+1}} > (c+1)(K(a, c))^{(\lambda+1)^e} > 0,$$

the resulting formula is equivalent to the formula \mathfrak{F}_4 . We now perform the same transformations on the formula thus obtained as we had carried out with respect to the formula \mathfrak{F}_4 in the proof of Theorem 1. As a result, we obtain a formula \mathfrak{F}_9 , which is analogous to formula (19).

We replace the formula \mathfrak{F}_5 by an equivalent formula

$$\begin{aligned} \exists z [b - (c+1)(K(a, c))^{(\lambda+1)^\delta} < 2(c+1)(K(a, c))^{(\lambda+1)^\delta} z < \\ (c+1)(K(a, c))^{(\lambda+1)^\delta} - b], \end{aligned}$$

which we denote by \mathfrak{F}_{10} .

The formula

$$\exists z \mathfrak{F}_7 \ \& \ \mathfrak{F}_8 \ \& \ \mathfrak{F}_9 \ \& \ \mathfrak{F}_{10} \tag{32}$$

is of a form analogous to (21). The only difference is as follows: the variables g_i, s_i, t_i were replaced in (32) by polynomials in the parameters a, b, c . It is easy to check that for all values of the parameters, the inequalities analogous to (20) hold. By Lemma 4 this enables us to find the desired polynomials F and W .

The theorem is proved.

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